Quantum form of Nonlinear Maxwell equations

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We generalize Maxwell equations which describe the vacuum of quantum electrodynamics into the quantum form. This nontraditional approach is different from the widely used theory—-Quantum Electrodynamics. From another viewpoint, it could be a direction for interpreting quantum theories properly.

I. INTRODUCTION

Since quantum electrodynamics is quite mature, it is not widely known that Maxwell equations are first quantized quantum equations of a photon after the Planck constant $\hbar$ was multiplied to it [15]. In fact, there has been an almost generally accepted way to change free noninteracting Maxwell equations into the form of Dirac equation[1]. The main purpose of this letter is to study Maxwell equations with self-interaction of a photon.

Much attention was paid to study spinor formulation of Maxwell equations after Dirac [2] find the relativistic equation for a particle with spin-1/2. Landau and Peierls photon wave functions [3] could ensure that the modulus squared of wave function be the probability density of a photon, but this approach has some weaknesses as Pauli [4] had pointed out. And Laporte and Uhlenbeck [5] gave Maxwell equations' spinor formulation

$$g_{\dot{1}\dot{1}} = 2(\Psi_2 + i\Psi_1),$$
$$g_{\dot{2}\dot{2}} = 2(\Psi_2 - i\Psi_1),$$
$$g_{\dot{1}\dot{2}} = g_{\dot{2}\dot{1}} = 2\Psi_3,$$

g_{ik} is a spinor of second rank and $\psi_j(\psi_j = E_j + iB_j)$ is a complex 3-vector wave function whose real and virtual parts are electric and magnetic fields. The most crucial step was made by Silberstein [6] who proved the possibility to describe Maxwell equations in terms of a 3-vector entities. However, it is Oppenheimer [7] who noticed quantum nature of Maxwell equations for the first time(there is other opinion about this [8]). And Good [9] gave more clear interpretation about the similarity between Maxwell equations and Dirac equation, the new Maxwell equations transformation properties and conservation theorems. Following the idea of Riemann-Silberstein-Majorana-Oppenheimer approach, Bialynicki-Birula [10–12] gave the more detailed consideration about single photon quantum mechanics in resent years.

In addition, Gersten [13–15] finds an alternative approach to gain the Dirac form of the Maxwell equations, which starts from the first principle and the derivation

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process is just like Dirac equation. If the momentum $\mathbf{p}$ is written in form of 3-vectors column

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad \mathbf{p}^{\mathrm{T}} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}.$$

Decomposing the left side of the relationship between energy and momentum of electrodynamic field after multiply a 3×3 unit matrix $\mathtt{I}$, we obtain

$$\left(\frac{E^2}{c^2} - \mathbf{p}^{\mathrm{T}}\mathbf{p}\right)\mathtt{I} = \left(\frac{E}{c}\mathtt{I} - \mathbf{p}^{\mathrm{T}}\mathbf{s}\right)\left(\frac{E}{c}\mathtt{I} + \mathbf{p}^{\mathrm{T}}\mathbf{s}\right) - \mathbf{p}\mathbf{p}^{\mathrm{T}} = 0,$$

where $\mathbf{s} = [s_1, s_2, s_3]^{\mathrm{T}}$ is a spin-1 vector matrix with three components

$$s_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, s_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (1)$$

Then the photon equation is obtained,

$$\left(\frac{E^2}{c^2} - \mathbf{p}^{\mathrm{T}}\mathbf{p}\right)\psi = \left(\frac{E}{c}\mathtt{I} - \mathbf{p}^{\mathrm{T}}\mathbf{s}\right)\left(\frac{E}{c}\mathtt{I} + \mathbf{p}^{\mathrm{T}}\mathbf{s}\right)\psi - \mathbf{p}\mathbf{p}^{\mathrm{T}}\psi = 0, (2)$$

where ψ is a 3 components wave function. At last, Eq. (2) holds for the following two expressions are satisfied

$$\left(\frac{E}{c}\mathtt{I} + \mathbf{p}^{\mathrm{T}} \cdot \mathbf{s}\right)\psi = 0, \tag{3}$$

$$\mathbf{p}^{\mathrm{T}} \cdot \psi = 0. \tag{4}$$

If the quantum operator substitutions and the wave function substitution are made as follows

$$E \Rightarrow i\hbar\frac{\partial}{\partial t}, \mathbf{p} \Rightarrow -i\hbar\bigtriangledown, \tag{5}$$

$$\psi = \mathbf{E} - i\mathbf{B}, \tag{6}$$

the classical Maxwell equations could be obtained

$$i\hbar\frac{\partial\psi}{\partial t} = -c\mathbf{s} \cdot \mathbf{p}\psi. \tag{7}$$

And the Gersten's approach return to Bialynicki-Birula's.

In addition, there are many works [16–27] concerning the Dirac-like form of Maxwell equations. In this paper, we extend linear quantum theory of Maxwell equations to

nonlinear situation caused by the possibility of creating virtual particles in vacuum. Starting from Lagrange with an corrections in higher orders in E and B, we obtained the nonlinear Maxwell equations, and then try to convert it into the nonlinear Schrödinger equation with the same approach and wave function as linear situation.

II. NONLINEAR SCHRÖDINGER FORM OF NONLINEAR MAXWELL EQUATIONS

The Lagrangian of Quantum electrodynamics in vacuum was given by Heisenberg and Euler[28]. It can describe the phenomenon of optical birefringence and experiments of measuring the vacuum birefringence.

In the limiting case of the stationary and homogeneous electromagnetic field, the exact expression of $\mathscr{L}$ would be

$$\mathscr{L} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{8}k_1(F_{\mu\nu}F^{\mu\nu})^2 + \frac{1}{8}k_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2, \quad (8)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, and $\tilde{F}^{\mu\nu}$ is the dual-field tensor contracted by $F_{\mu\nu}$ with completely antisymmetric unit tensor (the Levi-Civita tensor) $\epsilon^{\mu\nu\gamma\delta}$. $F_{\mu\nu}$ has the relation with A_μ as follow

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (9)$$

A_μ is vector potential in the electromagnetic field. so

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}, \quad (10)$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\gamma\delta}F_{\gamma\delta} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{bmatrix} \quad (11)$$

For notation here, Greek indices μ, ν, γ and δ run from 0 to 3.

And k_1, k_2 determine the magnitude of the nonlinear correction

$$k_1 = \frac{4\alpha^2}{45m_e^4}, \quad k_2 = \frac{7\alpha^2}{45m_e^4}, \quad (12)$$

α is fine-structure constant: $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$.

The last two terms of Eq. (8) are the corrections in $B^2 - E^2$ and $B \cdot E$ which are Lorentz invariants, so $\mathscr{L}$ is invariable in any frame of reference. $\mathscr{L}$ satisfies the following Lagrange's equation

$$\frac{\delta\mathscr{L}}{\delta A_\mu} - \partial_\mu\left(\frac{\delta\mathscr{L}}{\delta(\partial_\mu A_\nu)}\right) = 0. \quad (13)$$

If the Lagrange's equation holds for any situation, both terms must be zero,

$$\frac{\delta\mathscr{L}}{\delta A_\mu} = 0, \quad (14)$$

$$\partial_\mu\left(\frac{\delta\mathscr{L}}{\delta(\partial_\mu A_\nu)}\right) = 0. \quad (15)$$

Then put the $\mathscr{L}$ into Eq. (14) and Eq. (15), nonlinear Maxwell equations is obtained

$$\partial_t\mathbf{E}' = \nabla \times \mathbf{B}', \quad (16)$$
$$\partial_t\mathbf{B} = -\nabla \times \mathbf{E}, \quad (17)$$
$$\nabla \cdot \mathbf{E}' = 0, \quad (18)$$
$$\nabla \cdot \mathbf{B} = 0, \quad (19)$$

where

$$\mathbf{E}' = (1 - k_1x)\mathbf{E} - k_2y\mathbf{B}, \quad (20)$$
$$\mathbf{B}' = (1 - k_1x)\mathbf{B} + k_2y\mathbf{E}, \quad (21)$$
$$x = -2(\mathbf{E}^2 - \mathbf{B}^2), \quad (22)$$
$$y = -4\mathbf{E} \cdot \mathbf{B}. \quad (23)$$

Constructed with $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{E}'$ and $\mathbf{B}'$ take over their positions.

In order to get the Dirac form of nonlinear Maxwell equations, we combine Eq. (14) and Eq. (15)

$$\partial_t\begin{bmatrix} \mathbf{E}' \\ i\mathbf{B} \end{bmatrix} = \nabla \times \begin{bmatrix} \mathbf{B}' \\ -i\mathbf{E} \end{bmatrix}. \quad (24)$$

Substituting expressions (20) and (21) into Eq. (24), we separate $\mathbf{E}$ and $\mathbf{B}$ after partial differential operators ∂_t and ∇ act on $\mathbf{E}'$ and $\mathbf{B}'$

$$M\partial_t\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ iB_1 \\ iB_2 \\ iB_3 \end{bmatrix} = N\begin{bmatrix} iE_1 \\ iE_2 \\ iE_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad (25)$$

M and N are 6×6 matrices involving $\mathbf{E}$, $\mathbf{B}$ and space partial differential operators. See Matrix elements of M and N in Appendix.

In order to retain only time partial differential operator in the left side, both sides of Eq. (25) are multiplied by M^{-1},

$$\partial_t\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ iB_1 \\ iB_2 \\ iB_3 \end{bmatrix} = M^{-1}N\begin{bmatrix} iE_1 \\ iE_2 \\ iE_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad (26)$$

then $i\hbar$ times both sides of Eq. (26)

$$i\hbar\partial_t \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ iB_1 \\ iB_2 \\ iB_3 \end{bmatrix} = i\hbar M^{-1}N \begin{bmatrix} iE_1 \\ iE_2 \\ iE_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$= -\hbar M^{-1}N \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ iB_1 \\ iB_2 \\ iB_3 \end{bmatrix}. \quad (27)$$

If we assume

$$H = -\hbar M^{-1}N \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad (28)$$

and

$$\psi = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ iB_1 \\ iB_2 \\ iB_3 \end{bmatrix}, \quad (29)$$

then,

$$i\hbar\partial_t \psi = H\psi. \quad (30)$$

Regard Eq. (30) as the quantum equation of Maxwell equations including virtual processes (a photon the transform into a electron-positron pairs and interact with another photons).

We decompose H into two parts: Hamiltonian H_0 of classical electromagnetic fields and Hamiltonian H_{int} concluding virtual particles

$$H\psi = (H_0 + H_{int})\psi. \quad (31)$$

H_0 is almost identical to Eq. (16), which exists when the wave function has 6 components,

$$H_0 = -\hbar \begin{bmatrix} 0 & 0 & 0 & 0 & \partial_3 & -\partial_2 \\ 0 & 0 & 0 & -\partial_3 & 0 & \partial_1 \\ 0 & 0 & 0 & \partial_2 & -\partial_1 & 0 \\ 0 & \partial_3 & -\partial_2 & 0 & 0 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 & 0 & 0 \\ \partial_2 & -\partial_1 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (32)$$

H_0 can be expressed with $\mathbf{s}$ and $\mathbf{p}$

$$H_0 = \begin{bmatrix} 0 & \mathbf{s}\cdot\mathbf{p} \\ \mathbf{s}\cdot\mathbf{p} & 0 \end{bmatrix}. \quad (33)$$

p_k is momentum operater and Eq. (1) has given the exactly form of s_k. H_{int} could also be written in a compact manner

$$H_{int} = \Gamma^k \cdot p_k, \quad (34)$$

where Γ^k are 6×6 block matrices and consist of 3×3 matrixes $\mathcal{E}_k$ and $\mathcal{B}_k$ except zero elements

$$\Gamma^k = i \begin{bmatrix} -\mathcal{E}_k & \mathcal{B}_k \\ 0 & 0 \end{bmatrix}. \quad (35)$$

Elements of $\mathcal{E}_k$ and $\mathcal{B}_k$ are given in appendix and they satisfy the following relations

$$\mathcal{E}_k = \mathcal{E}_k^{\mathrm{T}}, \quad (36)$$
$$\mathcal{B}_k^{ij}(\mathbf{E},\mathbf{B}) = -\mathcal{B}_k^{ji}(\mathbf{B},\mathbf{E}), \quad (37)$$

where i,j,k run from 1 to 3. Eq. (37) means that, if we exchange $\mathbf{E}$ and $\mathbf{B}$ in the two elements which are symmetrical about the diagonal of $\mathcal{H}_{int}$ and reverse its sign, the two elements are equal.

In order to gain the same wave function as Eq. (6), transformation is made

$$i\hbar\partial_t U \begin{bmatrix} \mathbf{E} \\ i\mathbf{B} \end{bmatrix} = U H_0 U^{-1} U \begin{bmatrix} \mathbf{E} \\ i\mathbf{B} \end{bmatrix} + U H_{int} U^{-1} U \begin{bmatrix} \mathbf{E} \\ i\mathbf{B} \end{bmatrix} \quad (38)$$

where U is the transformation matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (39)$$

and U^{-1} is the inverse matrix of U. Then Eq.(38) can be written

$$i\hbar\partial_t \begin{bmatrix} \mathbf{E}+i\mathbf{B} \\ \mathbf{E}-i\mathbf{B} \end{bmatrix} = \mathcal{H}_0 \begin{bmatrix} \mathbf{E}+i\mathbf{B} \\ \mathbf{E}-i\mathbf{B} \end{bmatrix} + \mathcal{H}_{int} \begin{bmatrix} \mathbf{E}+i\mathbf{B} \\ \mathbf{E}-i\mathbf{B} \end{bmatrix}, \quad (40)$$

where $\mathcal{H}_0$ and $\mathcal{H}_{int}$ is the new Hamiltonian

$$\mathcal{H}_0 = U H_0 U^{-1} = \begin{bmatrix} \mathbf{s}\cdot\mathbf{p} & 0 \\ 0 & \mathbf{s}\cdot\mathbf{p} \end{bmatrix}, \quad (41)$$

$$\mathcal{H}_{int} = \frac{i}{2} \begin{bmatrix} -\mathcal{E}_k+\mathcal{B}_k & -\mathcal{E}_k-\mathcal{B}_k \\ -\mathcal{E}_k+\mathcal{B}_k & -\mathcal{E}_k-\mathcal{B}_k \end{bmatrix} \cdot p_k. \quad (42)$$

Obviously, the linear part of the Hamiltonian $\mathcal{H}_0$, which describes the classical electrodynamics, is a Hermitian operator. However, $\mathcal{H}_{int}$ is not a Hermitian operator although the relation between its elements shown in Eq. (36) and Eq. (37) is enlightening. Based on the Hamiltonian of Heisenberg and Euler which describes two photons scatter off one another in empty space(virtual particles are considered), we have got the Hamiltonian including $\mathcal{H}_{int}$. And the Hamiltonian is apparently not Hermitian. Accordingly, the number of photons is not conservative in the colliding process of two photons'.

III. CONCLUSION

In conclusion, starting from the Lagrangian $\mathscr{L}$ of quantum electrodynamics in vacuum, we give the corresponding Maxwell equations. Then a new approach which is different from the tradional method to derive from the form of nonlinear *Schrödinger* equation. At last, we transform the wave function in order to obtain Riemann-Silberstein-Majorana-Oppenheimer wave vector representation.

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Appendix A: Elements of $\mathcal{E}_k$ and $\mathcal{B}_k$

$$\mathcal{E}_1: \quad \mathcal{E}_1^{11} = 0, \tag{A1}$$
$$\mathcal{E}_1^{22} = -8k_1 E_2 B_3 + 8k_2 E_3 B_2,$$
$$\mathcal{E}_1^{33} = 8k_1 E_3 B_2 - 8k_2 E_2 B_3,$$
$$\mathcal{E}_1^{12} = \mathcal{E}_1^{21} = -4k_1 E_1 B_3 + 4k_2 E_3 B_1,$$
$$\mathcal{E}_1^{13} = \mathcal{E}_1^{31} = 4k_1 E_1 B_2 - 4k_2 E_2 B_1,$$
$$\mathcal{E}_1^{23} = \mathcal{E}_1^{32} = 4((k_1 - k_2)(E_2 B_2 - E_3 B_3).$$

$$\mathcal{E}_2: \quad \mathcal{E}_2^{11} = 8k_1 E_1 B_3 - 8k_2 E_3 B_1, \tag{A2}$$
$$\mathcal{E}_2^{22} = 0,$$
$$\mathcal{E}_2^{33} = -8k_1 E_3 B_1 - 8k_2 E_1 B_3,$$
$$\mathcal{E}_2^{12} = \mathcal{E}_2^{21} = 4k_1 E_2 B_3 - 4k_2 E_3 B_2,$$
$$\mathcal{E}_2^{13} = \mathcal{E}_2^{31} = 4(k_1 - k_2)(E_3 B_3 - E_1 B_1),$$
$$\mathcal{E}_2^{23} = \mathcal{E}_2^{32} = -4k_1 E_2 B_1 + 4k_2 E_1 B_2.$$

$$\mathcal{E}_3: \quad \mathcal{E}_3^{11} = -8k_1 E_1 B_2 + 8k_2 E_2 B_1, \tag{A3}$$
$$\mathcal{E}_3^{22} = 8k_1 E_2 B_1 - 8k_2 E_1 B_2,$$
$$\mathcal{E}_3^{33} = 0,$$
$$\mathcal{E}_3^{12} = \mathcal{E}_3^{21} = 4(k_1 - k_2)(E_1 B_1 - E_2 B_2),$$
$$\mathcal{E}_3^{13} = \mathcal{E}_3^{31} = -4k_1 E_3 B_2 + 4k_2 E_2 B_3,$$
$$\mathcal{E}_2^{23} = \mathcal{E}_2^{32} = 4k_1 E_3 B_1 - 4k_2 E_1 B_3.$$

$$\mathcal{B}_1: \quad \mathcal{B}_1^{11} = 0, \tag{A4}$$
$$\mathcal{B}_1^{22} = 4(k_1 - k_2)(B_2 B_3 - E_2 E_3),$$
$$\mathcal{B}_1^{33} = 4(k_1 - k_2)(E_2 E_3 - B_2 B_3)$$
$$\mathcal{B}_1^{12} = -4k_1 E_1 E_3 - 4k_2 B_1 B_3,$$
$$\mathcal{B}_1^{21} = 4k_1 B_1 B_3 + 4k_2 E_3 B_1,$$
$$\mathcal{B}_1^{13} = 4k_1 E_1 E_2 + 4k_2 B_1 B_2,$$
$$\mathcal{B}_1^{31} = -4k_1 B_1 B_2 - 4k_2 E_1 E_2,$$
$$\mathcal{B}_1^{23} = 4k_1 (E_2^2 + B_3^2) + 4k_2 (E_3^2 + B_2^2),$$
$$\mathcal{B}_1^{32} = -4k_1 (E_3^2 + B_2^2) - 4k_2 (E_2^2 + B_3^2).$$

$\mathcal{B}_2:$ $\mathcal{B}_2^{11} = 4(k_1 - k_2)(E_1 E_3 - B_1 B_3),$ (A5)

$\qquad \mathcal{B}_2^{22} = 0,$

$\qquad \mathcal{B}_2^{33} = -4(k_1 - k_2)(E_1 E_3 - B_1 B_3)$

$\qquad \mathcal{B}_2^{12} = -4k_1 B_2 B_3 - 4k_2 E_2 E_3,$

$\qquad \mathcal{B}_2^{21} = 4k_1 E_2 E_3 + 4k_2 B_2 B_3,$

$\qquad \mathcal{B}_2^{13} = -4k_1(E_1^2 + B_3^2) - 4k_2(E_3^2 + B_1^2),$

$\qquad \mathcal{B}_2^{31} = 4k_1(E_3^2 + B_1^2) + 4k_2(E_1^2 + B_3^2),$

$\qquad \mathcal{B}_2^{23} = -4k_1 E_1 E_2 - 4k_2 B_1 B_2,$

$\qquad \mathcal{B}_2^{32} = 4k_1 B_1 B_2 + 4k_2 E_1 E_2.$

$\mathcal{B}_3:$ $\mathcal{B}_3^{11} = -4(k_1 - k_2)(E_1 E_2 - B_1 B_2),$ (A6)

$\qquad \mathcal{B}_3^{22} = 4(k_1 - k_2)(E_1 E_2 - B_1 B_2),$

$\qquad \mathcal{B}_3^{33} = 0,$

$\qquad \mathcal{B}_3^{12} = 4k_1(E_1^2 + B_2^2) + 4k_2(E_2^2 + B_1^2),$

$\qquad \mathcal{B}_3^{21} = -4k_1(E_2^2 + B_1^2) - 4k_2(E_1^2 + B_2^2),$

$\qquad \mathcal{B}_2^{13} = 4k_1 B_2 B_3 + 4k_2 E_2 E_3,$

$\qquad \mathcal{B}_2^{31} = -4k_1 E_2 E_3 - 4k_2 B_2 B_3,$

$\qquad \mathcal{B}_2^{23} = -4k_1 B_1 B_3 - 4k_2 E_1 E_3,$

$\qquad \mathcal{B}_2^{32} = 4k_1 E_1 E_3 + 4k_2 B_1 B_3.$